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ON THE GENERAL SOLUTION FOR TORSION OF POLAR ELASTIC MEDIA

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Abstract—An approach is developed for determining general solutions for prescribed deformations in the strain-gradient theory of the polar elastic materials. Detailed solution is given for the pure torsion. Owing to the influence of the couple-stresses, the Poynting effect appears as a linear effect.

1. INTRODUCTION

EFFECTS of couple-stresses in elasticity attract since the papers of Günther in 1958 [1] and of Schäfer in 1962 [2] great attention. Most of the efforts, however, are concentrated on the investigations of these effects in the linear elasticity and on the concentration of stress around discontinuities (cracks, etc.) in a body. For references see [3]–[12].

The aim of this paper is to investigate the possibility of finding general solutions in Rivlin's sense [13], [14] for the non-linearized constitutive relations. Our considerations are restricted to the so-called polar elastic materials for which the internal energy is a function of the first and second order deformation gradients. The constitutive relations for such materials are derived by Grioli [15] and Toupin [16].

We shall use the notation of the two-point tensor fields. The points of a body in the initial (undeformed) configuration are referred to a system of material coordinates X^K with the metric tensor G_{KL}; the points of the deformed configuration of the body are referred to a system of spatial coordinates x^k with the metric tensor g_{kl}. A deformation is represented by the mappings

$$x^k = x^k(X^1, X^2, X^3); \tag{1.1}$$

$$X^K = X^K(x^1, x^2, x^3). \tag{1.2}$$

All capital latin indices will denote components of tensors with respect to the material frame of reference, and small latin indices will denote tensors with respect to the spatial frame of reference. The comma denotes the partial, and the semicolon the total covariant differentiation.

Under the assumption that the internal energy is a function of $x^k_{;K}$; and $x^k_{;KL}$; Toupin [17] derived the constitutive relations for a polar elastic material in the form

$$t^{(ij)} = \rho\left[g^{ik}\left(\frac{\partial\varepsilon}{\partial x^k_{;K}}x^j_{;K}+\frac{\partial\varepsilon}{\partial x^k_{;KL}}x^j_{;KL}\right)\right], \tag{1.3}$$

$$m^{i(jk)} = -2\rho g^{il}\frac{\partial\varepsilon}{\partial x^l_{;KL}}x^j_{;K}x^k_{;L}, \tag{1.4}$$

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where $t^{(ij)}$ is the symmetric part of the stress tensor t^{id}, $m^{i(jk)}$ is the symmetric part of the couple-stress tensor $m^{ijk} = -m^{jik}$, and ρ is the density of matter.

From the invariance requirement for the internal energy function under rigid body motions and from the symmetry properties of the couple-stress tensor Toupin obtained 13 partial differential equations with 27 variables $x^k_{;K}$, $x^k_{;KL}$,

$$\left[g^{ik}\left(\frac{\partial\varepsilon}{\partial x^k_{;K}}x^j_{;K}+\frac{\partial\varepsilon}{\partial x^k_{;KL}}x^j_{;KL}\right)\right]_{(ij)} = 0, \tag{1.5}$$

$$\left(g^{il}\frac{\partial\varepsilon}{\partial x^l_{;KL}}x^k_{;K}x^k_{;L}\right)_{(ijk)} = 0. \tag{1.6}$$

The internal energy is an arbitrary function of $27-13 = 14$ independent solutions of the system (1.5), (1.6). The solutions are

$$C_{KL} = g_{kl}x^k_{;K}x^l_{;L}, \tag{1.7}$$

$$D_{KLM} = \tfrac{2}{3}C_{M(K,L)} \tag{1.8}$$

and the constitutive equations can now be written in the form

$$t^{(ij)} = \rho\left(\frac{\partial\varepsilon}{\partial C_{KL}}x^i_{;K}x^j_{;L}+\frac{4}{3}\frac{\partial\varepsilon}{\partial D_{MKL}}x^{(i}_{;K}x^{j)}_{;LM}\right), \tag{1.9}$$

$$m^{i(jk)} = -\frac{4}{3}\rho\frac{\partial\varepsilon}{\partial D_{MKL}}x^i_{;K}x^{(j}_{;L}x^{k)}_{;M}. \tag{1.10}$$

The constitutive equations in this form are not suitable for the establishment of general solutions. In the next section we shall transform these equations to the spatial form. It is possible, in the so-obtained form of the constitutive relations, to eliminate the deformation gradients and to express the stress and the couple-stress tensors as explicit functions of certain measures of strain, but in that case stress and couple-stress are referred to the unit area of the initial state. In section 3 the constitutive relations are further transformed to a form in which the stress and couple-stress tensors are referred to the spatial system of reference, and the strain energy is a function of the invariants of the spatial measures of strain. The relations obtained will be used in section 4 for the analysis of torsion of a circular cylinder. The results obtained reduce under suitable conditions to the well-known results in the non-polar case but the Poynting effect, owing to the influence of couple-stresses, appears as an effect of the first order.

2. SPATIAL FORM OF THE CONSTITUTIVE RELATIONS

The constitutive relations (1.3), (1.4), or (1.9), (1.10) contain the material deformation gradients $x^k_{;K}$, $x^k_{;KL}$. To obtain those relations containing the spatial gradients we shall use the relations

$$X^A_{;a}x^b_{;A} = \delta^b_a, \tag{2.1}$$

$$x^j_{;KL} = -X^M_{;pq}x^p_{;K}x^q_{;L}x^j_{;M}. \tag{2.2}$$

From the chain-rule for differentiation,

$$\frac{\partial \varepsilon}{\partial x^k_{;K}} = \frac{\partial \varepsilon}{\partial X^A_{;a}} \frac{\partial X^A_{;a}}{\partial x^k_{;K}} + \frac{\partial \varepsilon}{\partial X^A_{;ab}} \frac{\partial X^A_{;ab}}{\partial x^k_{;K}}, \tag{2.3}$$

$$\frac{\partial \varepsilon}{\partial x^k_{;KL}} = \frac{\partial \varepsilon}{\partial X^A_{;an}} \frac{\partial X^A_{;ab}}{\partial x^k_{;KL}}, \tag{2.4}$$

we obtain from (1.3) and (1.4) the expressions

$$t^{(ij)} = -\rho g^{il}\left(\frac{\partial \varepsilon}{\partial X^K_{;j}} X^K_{;l} + 2\frac{\partial \varepsilon}{\partial X^{\ k}_{;jk}} X^K_{;kl}\right), \tag{2.5}$$

$$m^{i(jk)} = \rho g^{il} \frac{\partial \varepsilon}{\partial X^K_{;jk}} X^K_{;l}. \tag{2.6}$$

However, in the relations (2.5) and (2.6) the symmetry properties of $t^{(ij)}$ and $m^{i(jk)}$ are not preserved. The condition that the antisymmetric part of the right-hand side of (2.5) vanishes is equivalent to the requirement for the internal energy of the form

$$\varepsilon = \varepsilon(X^K_{;k}, X^K_{;kl}) \tag{2.7}$$

to be invariant under rigid-body motions (cf. Toupin [17])

$$\left[g^{il}\left(\frac{\partial \varepsilon}{\partial X^K_{;j}} X^K_{;l} + 2\frac{\partial \varepsilon}{\partial X^K_{;jk}} X^K_{;kl}\right)\right]_{(ij)} = 0. \tag{2.8}$$

From the antisymmetry of the couple-stress tensor, $m^{ijk} = -m^{jik}$, and from the symmetry of the left-hand side in (2.6) it follows that the right-hand side of that equation must satisfy the following ten conditions

$$\left[g^{il}\left(\frac{\partial \varepsilon}{\partial X^K_{;jk}} X^K_{;l}\right)\right]_{(ijk)} = 0. \tag{2.9}$$

The internal energy ε is now an arbitrary function of the independent solutions of 13 partial equations (2.8) and (2.9) with 27 independent variables $X^K_{;k}$ and $X^K_{;kl}$. The number of independent solutions is $27-13 = 14$, and the solutions are

$$\begin{aligned} \overset{-1}{C}{}^{MN} &\equiv g^{mn} X^M_{;m} X^N_{;n}, \\ R^{LMN} &\equiv \tfrac{2}{3} \overset{-1}{C}{}^{S[L} \overset{-1}{C}{}^{M]N}_{,S}. \end{aligned} \tag{2.10}$$

There are 36 such functions, but not all of them are independent, since they are satisfying $3+1+10+8 = 22$ relations of the form

$$\begin{aligned} &\overset{-1}{C}{}^{[MN]} = 0, \qquad R^{[LMN]} = 0, \qquad R^{(LMN)} = 0, \\ &R^{LMN} + R^{MLN} - R^{NML} - R^{LNM} = 0. \end{aligned} \tag{2.11}$$

The constitutive relations read now

$$T^{(AB)} = -2\rho_0\left(\frac{\partial \varepsilon}{\partial \overset{-1}{C}{}^{MN}} \overset{-1}{C}{}^{M(A} \overset{-1}{C}{}^{B)N} - \frac{4}{3}\frac{\partial \varepsilon}{\partial R^{M(LN)}} E^{ML(A} \overset{-1}{C}{}^{B)N}\right), \tag{2.12}$$

$$M^{A(BC)} = \tfrac{2}{3}\rho_0 \frac{\partial \varepsilon}{\partial R^{LMN}} \overset{-1}{C}{}^{MA} \overset{-1}{C}{}^{L(B} \overset{-1}{C}{}^{C)N}, \tag{2.13}$$

with

$$E^{MLA} = E^{MAL} = \tfrac{3}{2} R^{LMA} + \tfrac{1}{2} \overset{-1}{C}{}^{ML}_{;S} \overset{-1}{C}{}^{SA},$$

and

$$T^{AB} = \mathscr{J} X^A_{;i} X^B_{;j} t^{ij},$$

$$M^{ABC} = \mathscr{J} X^A_{;i} X^B_{;j} X^C_{;k} m^{ijk},$$

$$\mathscr{J} = \det x^k_{;K}.$$

It must be noted that the invariance condition (2.7) for the internal energy follows directly from (1.3) and therefore (2.8) does not represent a new restriction for ε. The same holds for (2.9) since it is a direct transform of the relation (1.4). Hence, the constitutive relations (2.12) and (2.13) are valid in general.

The expressions (2.12) and (2.13), however, are referred to the initial state coordinates, giving stress and couple-stress per unit area of the undeformed body. In order to obtain the components of stress and couple-stress in terms of spatial coordinates we shall assume that the material strain tensors **C** and **D** may be replaced in the internal energy function by the analogous spatial strain tensors **c** and **d**, defined by the expressions

$$\begin{aligned} c_{mn} &\equiv G_{MN} X^M_{;m} X^N_{;n}, \\ d_{mnr} &\equiv \tfrac{1}{3}(c_{rm,n} - c_{rn,m}) \equiv \tfrac{2}{3} c_{r[m,n]}. \end{aligned} \tag{2.14}$$

Obviously, the strain energy ε cannot, in general, be a function of **c** and **d**, except for some special classes of materials.

Writing

$$\frac{\partial \varepsilon}{\partial X^A_{;j}} = \frac{\partial \varepsilon}{\partial c_{mn}} \frac{\partial c_{mn}}{\partial X^A_{;j}} + \frac{\partial \varepsilon}{\partial d_{mnr}} \frac{\partial d_{mnr}}{\partial X^A_{;j}} \tag{2.15}$$

and

$$\frac{\partial \varepsilon}{\partial X^A_{;aj}} = \frac{\partial \varepsilon}{\partial d_{mnr}} \frac{\partial d_{mnr}}{\partial X^A_{;aj}}, \tag{2.16}$$

the stress–strain relations (2.5) can be transformed to the form

$$t^{(ij)} = \rho\left[g^{il}\left(2\frac{\partial \varepsilon}{\partial c_{jn}} c_{ln} + 2\frac{\partial \varepsilon}{\partial d_{jmn}} d_{lmn} + \frac{\partial \varepsilon}{\partial d_{mnj}} d_{mnl} \right)\right], \tag{2.17}$$

and the condition (2.8) obtains the form

$$\left[g^{il}\left(2\frac{\partial \varepsilon}{\partial c_{jn}} c_{ln} + 2\frac{\partial \varepsilon}{\partial d_{jmn}} d_{lmn} + \frac{\partial \varepsilon}{\partial d_{mnj}} d_{mnl} \right)\right]_{[ij]} = 0. \tag{2.18}$$

Similarly, expression (2.6) for the couple-stress tensor transforms to

$$m^{i(jk)} = \tfrac{1}{3}\rho g^{il}\left(\frac{\partial \varepsilon}{\partial d_{mkj}} + \frac{\partial \varepsilon}{\partial d_{mjk}} \right) c_{ml}, \tag{2.19}$$

and the symmetry condition (2.8) becomes

$$\left[g^{il}\left(\frac{\partial \varepsilon}{\partial d_{mjk}} + \frac{\partial \varepsilon}{\partial d_{mkj}} \right) c_{ml} \right]_{(ijk)} = 0. \tag{2.20}$$

The conditions (2.18) represent a set of three linear partial equations with $6+8 = 14$ independent variables c_{ij} and d_{ijk}. This set of equations admits $14-3 = 11$ independent solutions. But the ten equations (2.20) also represent restrictions on the arbitrariness of the strain energy function ε. The equations (2.20) admit only four independent solutions. Hence, the strain energy is an arbitrary function of only four simultaneous solutions of the equations (2.18) and (2.20).

If we introduce now the deviator $\mu_p^{\cdot k}$ of the second-order couple-stress tensor $m_p^{\cdot k}$,

$$m_p^{\cdot k} \equiv \tfrac{1}{2}e_{pij}m^{ijk}, \tag{2.21}$$

where e_{pij} is the totally antisymmetric Ricci tensor, we have

$$\mu_p^{\cdot k} = m_p^{\cdot k} - \tfrac{1}{3}m_I\delta_p^k, \qquad \mu_k^{\cdot k} \equiv 0$$
$$(m_I \equiv m_k^{\cdot k}) \tag{2.22}$$

and

$$\mu^{ijk} = e^{pij}\mu_p^{\cdot k} = m^{ijk} - \tfrac{1}{3}m_I e^{ijk}. \tag{2.23}$$

Evidently $\mu^{i(jk)} = m^{i(jk)}$ and using (2.22) we obtain the relation

$$\mu^{ijk} = \tfrac{2}{3}(2m^{i(jk)} + m^{k(ij)}), \tag{2.24}$$

so that the eight independent components of the third-order deviator of the couple-stress tensor may be directly expressed in terms of the spatial strain measures **c** and **d**.

3. CONSTITUTIVE RELATIONS FOR A CLASS OF POLAR ELASTIC MATERIALS

A dual representation of the strain tensor d_{mnr} is given by

$$d^l_{\cdot r} = \tfrac{1}{2}e^{lmn}d_{mnr} \tag{3.1}$$

and the strain energy ε may be considered as a function of c_{ij} and $d^l_{\cdot r}$.

The constitutive relations (2.17) for the symmetric part of the stress tensor have now the form

$$t^{(ij)} = -\rho g^{ip}\left(2\frac{\partial\varepsilon}{\partial c_{jn}}c_{np} + \frac{\partial\varepsilon}{\partial d^l_{\cdot j}}d^l_{\cdot p} + \frac{\partial\varepsilon}{\partial d^l_{\cdot r}}d^l_{\cdot t}\delta^j_p - \frac{\partial\varepsilon}{\partial d^p_{\cdot r}}d^j_{\cdot r}\right), \tag{3.2}$$

and the expression for the deviatoric part of the couple-stress tensor becomes

$$\mu_l^{\cdot k} = \frac{1}{9}\rho\frac{\partial\varepsilon}{\partial d^p_{\cdot m}}(c^k_l\delta^p_m + c^p_m\delta^k_l - c_I\delta^p_m\delta^k_l).$$
$$(c_I \equiv c^n_n) \tag{3.3}$$

The restrictive equations for the strain energy function ε reduce now to

$$\left[g^{ip}\left(2\frac{\partial\varepsilon}{\partial c_{jn}}c_{np} + \frac{\partial\varepsilon}{\partial d^l_{\cdot j}}d^l_{\cdot p} + \frac{\partial\varepsilon}{\partial d^l_{\cdot r}}d^l_{\cdot r}\delta^j_p - \frac{\partial\varepsilon}{\partial d^p_{\cdot r}}d^j_{\cdot r}\right)\right]_{[ij]} = 0, \tag{3.4}$$

$$\left[g^{ip}c_{mp}\left(\frac{\partial\varepsilon}{\partial d^l_{\cdot j}}e^{lkn} + \frac{\partial\varepsilon}{\partial d^l_{\cdot k}}e^{ljn}\right)\right]_{(ijk)} = 0. \tag{3.5}$$

The four solutions of (3.4) which satisfy (3.5) are: three basic invariants of the deformation tensor **c** and one mixed (or joint) invariant of the tensors **c** and **d**,

$$\mathrm{I}_{\mathbf{c}} = \delta^{a}_{b}c^{b}_{a}, \qquad \mathrm{II}_{\mathbf{c}} = \tfrac{1}{2}\delta^{ab}_{pq}c^{p}_{a}c^{q}_{b}$$
$$\mathrm{III}_{\mathbf{c}} = \tfrac{1}{6}\delta^{abc}_{pqr}c^{p}_{a}c^{q}_{b}c^{r}_{c}, \qquad \mathrm{II}_{m} = c^{a}_{b}d^{b}_{.a}. \tag{3.6}$$

Accordingly, ε is now an arbitrary function of the invariants (3.7), and the constitutive relations (3.2) and (3.3) become

$$t^{(ij)} = \overset{*}{t}{}^{(ij)} - \rho\frac{\partial\varepsilon}{\partial \mathrm{II}_m}(\mathrm{II}_m g^{ij} + c^{i}_{n}d^{nj} + c^{j}_{n}d^{ni}), \tag{3.7}$$

$$\mu_{l}^{\cdot k} = \frac{1}{9}\rho\frac{\partial\varepsilon}{\partial \mathrm{II}_m}(3c^{n}_{l}c^{k}_{n} - 3\mathrm{I}_{\mathbf{c}}c^{k}_{l} + 2\mathrm{II}_{\mathbf{c}}\delta^{k}_{l}). \tag{3.8}$$

Here, $\overset{*}{\mathbf{t}}$ is the stress tensor which appears in the non-polar theory,

$$\overset{*}{t}{}^{(ij)} = -2\rho c^{i}_{n}\left(\frac{\partial\varepsilon}{\partial \mathrm{I}_{\mathbf{c}}}\frac{\partial \mathrm{I}_{\mathbf{c}}}{\partial c_{jn}} + \frac{\partial\varepsilon}{\partial \mathrm{II}_{\mathbf{c}}}\frac{\partial \mathrm{II}_{\mathbf{c}}}{\partial c_{jn}} + \frac{\partial\varepsilon}{\partial \mathrm{III}_{\mathbf{c}}}\frac{\partial \mathrm{III}_{\mathbf{c}}}{\partial c_{jn}}\right). \tag{3.9}$$

The invariance group of the strain energy function ε, [which is a function of the invariants (3.7)], and of the constitutive relations (3.8), (3.9), is the proper orthogonal group [18]. Thus, the material whose elastic response is described by the constitutive equations (3.8) and (3.9) is the isotropic material with no center of symmetry.

4. APPLICATION: TORSION OF AN INCOMPRESSIBLE HOMOGENEOUS CIRCULAR CYLINDER

In the theory of non-polar hyperelastic materials, torsion of a circular cylinder is one of the most impressive examples of Rivlin's [13], [14] general solutions. In the theory of polar hyperelastic materials, gradients of vorticity are the sources of couple-stresses and torsion is one of the simplest deformations in which one may study the influence of couple-stresses and compare the results with those obtained by Rivlin in the non-polar case.

Our treatment of the problem is analogous to Truesdell's [19] exposition of Rivlin's work.

Let spatial and material coordinates be cylindrical coordinates,

$$\{x^1, x^2, x^3\} = \{r, \vartheta, z\}, \tag{4.1}$$

$$\{X^1, X^2, X^3\} = \{R, \Theta, Z\}. \tag{4.2}$$

We are regarding the torsion of a full homogeneous cylinder of radius a. The deformation is given by

$$r = R, \qquad \vartheta = \Theta + KZ, \qquad z = Z, \tag{4.3}$$

or

$$R = r, \qquad \Theta = \vartheta - Kz, \qquad Z = z. \tag{4.4}$$

The mixed deformation tensor **c** is represented by the matrix

$$\{c^i_j\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & -K \\ 0 & -r^2K & 1+r^2K \end{Bmatrix} \tag{4.5}$$

and its principal invariants are

$$\mathrm{I}_c = \mathrm{II}_c = 3+r^2K^2, \qquad \mathrm{III}_c = 1. \tag{4.6}$$

After the covariant differentiation of the covariant components of **c**, from (2.14) and (3.1) we obtain the components of the mixed second-order tensor $d^k_{\cdot j}$:

$$\{d^k_j\} = \begin{Bmatrix} \frac{1}{3}rK & 0 & 0 \\ 0 & \frac{1}{3}rK & -\frac{2}{3}rK^2 \\ 0 & 0 & -\frac{2}{3}rK^2 \end{Bmatrix}, \tag{4.7}$$

and from $(3.7)_4$, (4.5) and (4.7) it follows that

$$\mathrm{II}_m = 0. \tag{4.8}$$

Now from (3.8) we obtain directly the components of the symmetric part of the stress tensor, which in matrix form reads

$$\{t^{(ij)}\} = \{\overset{*}{t}{}^{(ij)}\} + \frac{\partial\varepsilon}{\partial \mathrm{II}_m}\begin{Bmatrix} -\frac{2}{3}rK & 0 & 0 \\ 0 & -\frac{2}{3}\frac{K}{r} & \frac{1}{3}rK^2 \\ 0 & \frac{1}{3}rK^2 & \frac{4}{3}rK \end{Bmatrix}, \tag{4.9}$$

and for the deviatoric part of the couple-stress tensor we have from (3.9)

$$\{\mu_l^{\cdot k}\} = \frac{1}{9}\frac{\partial\varepsilon}{\partial \mathrm{II}_m}\begin{Bmatrix} -r^2K^2 & 0 & 0 \\ 0 & 2r^2K^2 & 3K \\ 0 & 3r^2K^2 & -r^2K^2 \end{Bmatrix}. \tag{4.10}$$

The part $\overset{*}{t}{}^{(ij)}$ of the stress tensor coincides with the components of stress in the non-polar case (p is the hydrostatic pressure),

$$\begin{aligned} \overset{*}{t}{}^{11} &= -p+2\left(\frac{\partial\varepsilon}{\partial \mathrm{I}_c}-\frac{\partial\varepsilon}{\partial \mathrm{II}_c}\right), & \overset{*}{t}{}^{(12)} &= 0, \quad \overset{*}{t}{}^{(13)} = 0, \\ \overset{*}{t}{}^{22} &= \frac{1}{r^2}\left(\overset{*}{t}{}^{11}+2r^2K^2\frac{\partial\varepsilon}{\partial \mathrm{I}_c}\right), & \overset{*}{t}{}^{(23)} &= 2K\left(\frac{\partial\varepsilon}{\partial \mathrm{I}_c}+\frac{\partial\varepsilon}{\partial \mathrm{II}_c}\right), \\ \overset{*}{t}{}^{33} &= \overset{*}{t}{}^{11}-2r^2K^2\frac{\partial\varepsilon}{\partial \mathrm{II}_c}. \end{aligned} \tag{4.11}$$

Comparing the components of stress $t^{(ij)}$ in the polar case with the components of stress in the non-polar case, we see that the differences are only quantitative.

The equilibrium conditions in the absence of volume forces and couples reduce to the equations

$$t^{ij}_{,j} = t^{(ij)}_{,j} + t^{[ij]}_{,j} = 0. \tag{4.12}$$

Since the antisymmetric part of the stress tensor is connected with the couple-stress tensor by the relation

$$t^{[ij]} = m^{ijk}_{,k},$$

it follows that

$$t^{[ij]}_{,j} = m^{ijk}_{,kj} = m^{i(jk)}_{,kj}$$

and from (2.22) we have

$$t^{[ij]}_{,j} = \mu^{i(jk)}_{,jk}.$$

Hence, the constitutive relations (4.10) are sufficient for the determination of the divergence $t^{[ij]}_{,j}$ of the non-symmetric part of the stress tensor. From (4.10) we easily find that

$$\mu^{i(jk)}_{,jk} = 0$$

for all $i = 1, 2, 3$. The equilibrium equations (4.12) reduce now to the form which contains only the symmetric components of the stress tensor. Since all invariants $\mathrm{I_c}, \mathrm{II_c}, \mathrm{II}_m$ are functions of r only, and $\mathrm{III_c} = 1$, the strain energy is a function of r only and the equilibrium equations reduce to

$$\begin{aligned} &\frac{\partial}{\partial r}\left(\overset{*}{t}{}^{11} - \frac{2}{3} r K \frac{\partial \varepsilon}{\partial \mathrm{II}_m}\right) - 2rK^2 \frac{\partial \varepsilon}{\partial \mathrm{I_c}} = 0, \\ &\frac{\partial \overset{*}{t}{}^{11}}{\partial \vartheta} = 0, \qquad \frac{\partial \overset{*}{t}{}^{11}}{\partial z} = 0, \end{aligned} \tag{4.13}$$

and we have

$$\overset{*}{t}{}^{11} = \frac{2}{3} r K \frac{\partial \varepsilon}{\partial \mathrm{II}_m} + 2K^2 \int_a^r r \frac{\partial \varepsilon}{\partial \mathrm{I_c}}\, \mathrm{d}r, \tag{4.14}$$

if the surface $r = a$ of the cylinder is to be free of traction. In the non-polar case $(\partial \varepsilon / \partial \mathrm{II}_m) = 0$ and (4.14) obtains the usual form.

The normal force which must be applied at the ends of the cylinder to prevent the dilatation or contraction of the cylinder when twisted is

$$N = 2\pi \int_0^a t^{33} r \, \mathrm{d}r. \tag{4.15}$$

Using (4.9), (4.11) and (4.14) we obtain

$$N = 2\pi \int_0^a \left[\frac{1}{2} K \frac{\partial \varepsilon}{\partial \mathrm{II}_m} r + 2K^2 \left(\int_a^r r \frac{\partial \varepsilon}{\partial \mathrm{I_c}}\, \mathrm{d}r - r^2 \frac{\partial \varepsilon}{\partial \mathrm{II_c}}\right)\right] r \, \mathrm{d}r, \tag{4.16}$$

which is equivalent to

$$N = \overset{*}{N} + \pi K \int_0^a r^2 \frac{\partial \varepsilon}{\partial \mathrm{II}_m} \mathrm{d}r, \tag{4.17}$$

where

$$\overset{*}{N} = -\pi K^2 \int_0^a r^2 \left(\frac{\partial \varepsilon}{\partial \mathrm{I}_c} + 2\frac{\partial \varepsilon}{\partial \mathrm{II}_c} \right) \mathrm{d}r^2 \tag{4.18}$$

is the normal force in the non-polar case.

From (4.17) it follows that in the polar case the force that must be applied to the ends of the cylinder in order to prevent its dilatation (or contraction) has a term linear in K, and in the non-polar case the lowest power of K is two. Hence, in the case of polar isotropic elastic materials the Poynting effect is not an effect of the second order, but of the first order. For small twists the internal energy ε may be approximated by a quadratic polynomial

$$\varepsilon = \alpha \mathrm{I}_c^2 + \beta \mathrm{II}_c + \gamma \mathrm{II}_m,$$

and the expression (4.16) gives for the normal force

$$N = K\frac{\pi \gamma a^3}{3} - K^2 \pi (3\alpha + \beta) a^4 - K^4 \frac{2\pi \alpha a^6}{3},$$

which in the first approximation reduces to the linear relation

$$N \approx K\frac{\pi \gamma a^3}{3}$$

in which γ is an unknown coefficient.

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Абстракт—Предлагается способ для определения общих решений для заданных деформаций в теории градиента деформации полярноупругих материалов. Приводится подробное решение для случая чистого кручения. Эффект Пойнтинга оказывается линейным эффектом вследствие влияния моментных напряжений.